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Bianchi type IX non-static barotropic perfect fluid cosmological model in general relativity

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Abstract

Bianchi type IX barotropic perfect fluid cosmological models are investigated assuming that shear (σ) is proportional to the expansion (θ). This leads to $a = b^n$ where a and b are metric potentials and n is the constant. For the realistic picture of the universe in terms of cosmic time t, $n = 2$ and 3/2. The physical and geometrical aspects of the models are discussed in detail

(Keywords : Bianchi IX/non-static/barotropic/perfect fluid/ cosmological)

Introduction

Bianchi type IX cosmological models are the generalization of Friedmann-Robertson-Walker (FRW) model with positive curvature. Bianchi type IX string cosmological models are investigated by Chakraborty and Nandy[1], Chakraborty[2], Bali and Dave[3], Bali and Upadhaya[4]. Bali and Yadav[5] have investigated Bianchi type IX viscous fluid cosmological model in general relativity. To get the deterministic solution they have assumed that the coefficient of shear viscosity (η) is proportional to the expansion (θ). Bali and Upadhaya[6] have obtained Bianchi type IX string cosmological models with bulk viscosity. Bali *et al.*[7] have investigated Bianchi type IX inflationary universe in general relativity.

In this paper, we have investigated Bianchi type IX barotropic perfect fluid cosmological models in general relativity assuming that the shear (σ) is proportional to the expansion (θ). This leads to $a = b^n$ where a and b are metric potentials and n is the constant. Here we have assumed the proportionality constant as unity. To get the realistic picture of the universe in terms of cosmic time t, we have assumed two cases for $n = 2$ and 3/2. Both the models represent realistic models. The physical and geometrical aspects of the models are discussed in detail.

The Metric and Field Equations

We consider the Bianchi type IX metric in the form

$$ds = -dt^2 + a^2(t)\,dx^2 + b^2(t)\,dy^2 + (b^2\sin^2 y + a^2\cos^2 y)\,dz^2 - 2a^2\cos y\,dxdy \quad (1)$$

where a and b are functions of t alone.

The energy momentum tensor (T_i^j) for perfect fluid distribution is given by

$$T_i^j = (\in + p)v_i v^j + p g_i^j \quad (2)$$

where $\in$ is the energy density, p the isotropic pressure and v^i the flow velocity satisfying

$$g_{ij}\,v^i v^j = -1.$$

We assume that coordinates are comoving so that

$$v^1 = 0 = v^2 = v^3,\; v^4 = 1$$

Einstein field equation

$$R_i^j - \frac{1}{2}Rg_i^j = -8\pi T_i^j$$

for the metric (1) leads to

$$\frac{2\ddot{b}}{b}+\frac{\dot{b}^2}{b^2}+\frac{1}{b^2}-\frac{3a^2}{4b^4}=-8\pi p \tag{3}$$

$$\frac{\ddot{b}}{b}+\frac{\ddot{a}}{a}+\frac{\dot{a}\dot{b}}{ab}+\frac{a^2}{4b^4}=-8\pi p \tag{4}$$

$$\frac{2\dot{a}\dot{b}}{ab}+\frac{\dot{b}^2}{b^2}+\frac{1}{b^2}-\frac{a^2}{4b^4}=-8\pi\varepsilon. \tag{5}$$

Solution of Field Equations

Equations (3) and equation (4) lead to

$$\frac{\ddot{b}}{b}+\frac{\dot{b}^2}{b^2}+\frac{1}{b^2}-\frac{\ddot{a}}{a}-\frac{\dot{a}\dot{b}}{ab}-\frac{a^2}{b^4}=0 \tag{6}$$

Equations (3) to (5) are three equations in four unknowns a, b, $\in$ and p. To get the deter-ministic model, we assume that the model is filled with barotropic perfect fluid distribution which leads to $p = \gamma \in$, where $0 \leq \gamma \leq 1$. Equations (3) and (5) after using condition $p = \gamma\in$, lead to

$$\frac{2}{\gamma}\frac{\ddot{b}}{b}+\left(\frac{1+\gamma}{\gamma}\right)\left(\frac{\dot{b}^2}{b^2}+\frac{1}{b^2}\right)-\left(\frac{3+\gamma}{\gamma}\right)\frac{a^2}{4b^4}+\frac{2\dot{a}\dot{b}}{ab}=0 \tag{7}$$

To get the deterministic solution, we also assume that $\sigma \propto \theta$ which leads to

$$a = b^n \tag{8}$$

Thus we have

$$\frac{\dot{a}}{a}=n\frac{\dot{b}}{b} \tag{9}$$

Using equations (8) and (9) in equation (7), we have

$$\frac{2\ddot{b}}{b}+(1+\gamma+2n\gamma)\frac{\dot{b}^2}{b^2}-(3+\gamma)\frac{b^{2n-4}}{4}+(1+\gamma)\frac{1}{b^2}=0 \tag{10}$$

which again leads to

$$2\ddot{b}+(1+\gamma+2n\gamma)\frac{\dot{b}^2}{b}-(3+\gamma)\frac{b^{2n-3}}{4}+(1+\gamma)\frac{1}{b}=0 \tag{11}$$

and

$$f=\left[Ab^{2n-2}-B\right]^{1/2} \tag{12}$$

where $\dot{b}=f(b)$, $\ddot{b}=ff'$ and $f'=\frac{df}{db}$.

Thus

$$A=\frac{(3+\gamma)}{4(2n+\gamma+2n\gamma-1)} \tag{13}$$

$$B=\frac{(1+\gamma)}{(1+2n\gamma+\gamma)} \tag{14}$$

To get the deterministic value of b, we assume two different cases of $n = 2$ and $3/2$.

The First Model

We assume $n = 2$, then equation (12) leads to

$$f = [Ab^2 - B]^{1/2} \tag{15}$$

which leads to

$$\frac{db}{\sqrt{b^2-\beta^2}}=\sqrt{A}dt \tag{16}$$

where

$$\beta^2=\frac{B}{A} \tag{17}$$

On integration equation (16) leads to

$$b=\beta\cosh\left(\sqrt{A}t+R\right) \tag{18}$$

where R is the constant of integration.

Using equation (18) in equation (8), we have

$$a=\beta^2\left[\cosh\left(\sqrt{A}t+R\right)\right]^2 \tag{19}$$

Thus equations (13), (14) and (17) after using n = 2 reduces to the form

$$A = \frac{3+\gamma}{4(3+5\gamma)} \tag{20}$$

$$B = \frac{1+\gamma}{1+5\gamma} \tag{21}$$

$$\beta^2 = \frac{B}{A} = \frac{4(1+\gamma)(3+5\gamma)}{(1+5\gamma)(3+\gamma)} \tag{22}$$

Hence the metric (1) leads to

$$\begin{aligned} ds^2 &= -dt^2 \\ &+\beta^2 \cosh^2\left(\sqrt{A}t + R\right)\left\{\beta^2 \cosh^2\left(\sqrt{A}t + R\right)dx^2 + dy^2\right\} \\ &+\beta^2 \cosh^2\left(\sqrt{A}t + R\right) \\ &\left\{\sinh^2 y + \beta^2 \cosh^2\left(\sqrt{A}t + R\right)\cos^2 y\right\}dz^2 \\ &-2\beta^4 \cosh^4\left(\sqrt{A}t + R\right)\cos y\, dx dy \end{aligned} \tag{23}$$

Using the transformations

$$x = X, \quad y = Y, \quad z = Z$$

and

$$\left(\sqrt{A}t + R\right) = T$$

the metric (23) reduces to the form

$$\begin{aligned} ds^2 &= -\frac{dT^2}{A} + \beta^2 \cosh^2 T\left[\beta^2 \cosh^2 T dX^2 + dY^2\right] \\ &+\beta^2 \cosh^2 T\left[\sinh^2 Y + \beta^2 \cosh^2 T \cos^2 Y\right]dZ^2 \\ &-2\beta^2 \cosh^4 T \cos Y\, dX\, dY \end{aligned} \tag{24}$$

Some Physical and Geometrical Features

The energy density ($\in$), the isotropic pressure (p) for the metric (24) are given by

$$8\pi \in = 5A \tanh^2 T + \frac{1}{\beta^2}\operatorname{sec h}^2 T - \frac{1}{4} \tag{25}$$

and

$$8\pi p = \frac{3}{4} - 2A - A\tanh^2 T - \frac{1}{\beta^2}\operatorname{sech}^2 T \tag{26}$$

The expansion (θ) and the components of shear tensor (σ_i^j) are given by

$$\theta = 4\sqrt{A}\tanh T \tag{27}$$

$$\sigma_1^1 = \frac{2}{3}\sqrt{A}\tanh T \tag{28}$$

$$\sigma_2^2 = -\frac{1}{3}\sqrt{A}\tanh T \tag{29}$$

$$\sigma_3^3 = -\frac{1}{3}\sqrt{A}\tanh T \tag{30}$$

$$\sigma_4^4 = 0 \tag{31}$$

Thus

$$\sigma^2 = \frac{1}{2}\left[\left(\sigma_1^1\right)^2 + \left(\sigma_2^2\right)^2 + \left(\sigma_3^3\right)^2 + \left(\sigma_4^4\right)^2\right]$$

Therefore

$$\sigma = \sqrt{\frac{A}{3}}\tanh T \tag{32}$$

The ratio $\frac{\sigma}{\theta}$ is given by

$$\sigma = \frac{1}{4\sqrt{3}}\theta.$$

The spatial volume (R^3) and the deceleration parameter (q) are given by

$$R^3 = \beta^4 \cosh^4 T \sin y \tag{33}$$

$$q = -\left[1 + \frac{3}{4\sinh^2 T}\right] \tag{34}$$

Discussion

The reality conditions (i) $\in + p > 0$, (ii) $\in + 3p > 0$ given by Ellis[8] are satisfied when $\tan h^2 T > \frac{4A-1}{8A}$. The expansion in the model (24) increases as time increases. When $T \to \infty$, then $\theta \to 4\sqrt{A}$. When $T \to 0$, then $\sigma \to 0$ and when $T \to \infty$, then $\sigma \to \sqrt{A}/3$. The spatial volume increases as time increases. Since the deceleration parameter $q < 0$, hence the model (24) represents an accelerating universe.

The Second Model

We assume that $n = 3/2$, then equation (12) leads to

$$f = (Ab - B)^{1/2} \tag{35}$$

which leads to

$$\frac{db}{\sqrt{b-\alpha}} = dt \tag{36}$$

where

$$\alpha = \frac{B}{A}. \tag{37}$$

On integration equation (36) leads to

$$b = 4\left(\sqrt{At} + m\right)^2 + \alpha \tag{38}$$

Using equation (38) in equation (8), we have

$$a = \left[4\left(\sqrt{At} + m\right)^2 + \alpha\right]^{3/2}. \tag{39}$$

Thus equations (13), (14) and (37) after using the assumption $n = 3/2$, leads to

$$A = \frac{3+\gamma}{8(1+2\gamma)} \tag{40}$$

$$B = \frac{1+\gamma}{1+4\gamma} \tag{41}$$

$$\alpha = \frac{8(1+\gamma)(1+2\gamma)}{(1+4\gamma)(3+\gamma)}. \tag{42}$$

Hence the metric (1) leads to

$$ds^2 = -dt^2 + \left[4\left(\sqrt{At} + m\right)^2 + \alpha\right]^3 dx^2 + \left[4\left(\sqrt{At} + m\right)^2 + \alpha\right]^2 dy^2 + \left[4\left(\sqrt{At} + m\right)^2 + \alpha\right]^2 \left[\sin^2 y + \left\{4\left(\sqrt{At} + m\right)^2 + \alpha\right\}\cos^2 y\right]dz^2$$

$$-2\left[4\left(\sqrt{At} + m\right)^2 + \alpha\right]^3 \cos y \, dx\, dz \tag{43}$$

After using transformations

$$x = X, \quad y = Y, \quad z = Z$$

and

$$\left(\sqrt{At} + m\right) = \tau$$

equation (43) leads to

$$ds^2 = -\frac{d\tau^2}{A} + (4\tau^2 + \alpha)^3 dX^2 + (4\tau^2 + \alpha)^2 dY^2$$

$$+(4\tau^2 + \alpha)^2\left[\sin^2 Y + (4\tau^2 + \alpha)\cos^2 Y\right]dZ^2$$

$$-2(4\tau^2 + \alpha)^3 \cos Y \, dX\, dZ \tag{44}$$

Some Physical and Geometrical Features

The energy density ($\in$), the isotropic pressure (p) for the metric (44) are given by

$$8\pi \in = \frac{4(256A - 1)\tau^2 + 4 - \alpha}{4\left[4\tau^2 + \alpha\right]^2} \tag{45}$$

$$8\pi p = \frac{4(3 - 128A)\tau^2 + (3 - 64A)\alpha - 4}{4\left[4\tau^2 + \alpha\right]^2}. \tag{46}$$

The expansion (θ) and the components of shear tensor (σ_i^j) are given by

$$\theta = \frac{28\sqrt{A}}{\tau\left(\frac{4}{\sqrt{A}} + \frac{\alpha}{\tau^2}\right)} \tag{47}$$

$$\sigma_1^1 = -\frac{8\sqrt{A}\tau}{3(4\tau^2 + \alpha)} \tag{48}$$

$$\sigma_2^2 = -\frac{4}{3}\frac{\sqrt{A}\tau}{\left(4\tau^2+\alpha\right)} \tag{49}$$

$$\sigma_3^3 = -\frac{4}{3}\frac{\sqrt{A}\tau}{\left(4\tau^2+\alpha\right)} \tag{50}$$

$$\sigma_4^4 = 0. \tag{51}$$

Thus the shear tensor (σ) is given by

$$\sigma = \frac{4\sqrt{A}\tau}{\sqrt{3}\left(4\tau^2+\alpha\right)} \tag{52}$$

and the ratio

$$\frac{\sigma}{\theta} = \frac{1}{7\sqrt{3}}.$$

The spatial volume (R^3) and the deceleration parameter (q) are given by

$$R^3 = \left[4\tau^2+\alpha\right]^{7/2} \sin Y \tag{53}$$

$$q = -\frac{\left[16\tau^2+3\alpha\right]}{28\tau^2} \tag{54}$$

Discussion

The reality conditions (i) $\in + p > 0$, (ii) $\in + 3p > 0$ given by Ellis[8] are satisfied during the span of time given by

$$-\sqrt{\frac{64A\alpha-2\alpha}{512A+8}} < \tau < \sqrt{\frac{64A\alpha-2\alpha}{512A+8}}. \tag{55}$$

The model (44) starts with a big-bang at $\tau = 0$ and the expansion in the model decreases as time increases. Since $\sigma = 0$ when $\tau = \infty$, hence the model isotropizes for large value of τ. Also $\frac{\sigma}{\theta}$ is constant throughout. The spatial volume (R^3) increases as time increases. Since the deceleration parameter $q < 0$, hence the model represents an accelerating universe during the span of time given by expression (55).

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