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Bianchi type I static barotropic perfect fluid model in general relativity

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Abstract. Bianchi type I static space-time filled with barotropic perfect fluid distribution ($p = \gamma\, \rho$) in general relativity, is investigated where p is the isotropic pressure, ρ the energy density and $0 \leq \gamma \leq 1$. To get the deterministic model of the universe, we have also considered some special cases which fulfills the barotropic equation of state. Some other physical aspects of the model, are also discussed.

Keywords. Bianchi I, static, barotropic, perfect fluid.

1. Introduction

Static universes play a significant role in understanding the phenomena of cosmological and astrophysical significance. Einstein and de-Sitter derived static cosmological models in which the material distribution is that of perfect fluid. Static cylindrically symmetric space-time representing material distribution has been obtained by Marder [6]. Hargreaves [5] has discussed the stability of static spherically symmetric fluid spheres which consist of a core of ideal gas and radiation where the ratio of gas pressure and the total pressure is a small constant. Singh and Abdussattar [8] have obtained an exact static spherically symmeric solution of Einstein field equation for disordered radiation and to overcome the difficulty of infinite density at the centre, they have assumed that the distribution has a core of finite radius r_0 and constant density ε_0 which is fitted to the solution of disordered radiation. Teixeira, Wolk and Som [9] have obtained an exact solution of physically analogous systems with plane symmetry. Teixeira, Wolk and Som [10] have also considered a source free disordered distribution of electromagnetic radiation and obtained a time independent exact solution with cylindrical symmetry. Roy and Bali [7] have investigated some magnetohydrostatic models of cylindrical symmery in which the free gravitational field is Petrov type ID, the degeneracy being in the xy-plane. Bali and Jain [1], Bali and Tyagi [3] have investigated some magnetostatic perfect fluid cosmological models filled with dust, disordered radiation and stiff fluid. Recently Bali and Jain [2] have investigated Bianchi type I static stiff perfect fluid magnetized cosmological model in General Relativity.

In this paper, we have investigated static space-time filled with barotropic perfect fluid distribution using the condition on metric potentials. To get the deterministic model of the universe, we have also considered some special cases which fulfills the barotropic equation of state. The model in general represents Petrov type ID and for large value of x, the model represents conformally flat space-time. Some other physical aspects of the model are also discussed.

2. The matric and field equations

We consider the Bianchi type I metric in the form given by Marder [6] as

$$ds^2 = A^2 (dx^2 - dt^2) + B^2 dy^2 + C^2 dz^2 \tag{1}$$

where A, B, C are functions of x-alone.

The energy momentum tensor for perfect fluid is given by

$$T_i^j = (\varepsilon + p)\, v_i v^j + p\, g_i^j \tag{2}$$

where ε is the energy density, p the isotropic pressure, and v^i the flow velocity satisfying

$$g_{ij} v^i v^j = -1$$

We assume the coordinates to be comoving so that

$$v^1 = 0 = v^2 = v^3 \text{ and } v^4 = \frac{1}{A}$$

The Einstein field equation

$$R_i^j - \frac{1}{2} R g_i^j = -8\pi T_i^j \tag{3}$$

for metric (1) leads to

$$\frac{1}{A^2}\left[\frac{A_1 B_1}{AB} + \frac{A_1 C_1}{AC} + \frac{B_1 C_1}{BC}\right] = 8\pi p \tag{4}$$

$$\frac{1}{A^2}\left[\frac{C_{11}}{C} + \frac{A_{11}}{A} - \frac{A_1^2}{A^2}\right] = 8\pi p \tag{5}$$

$$\frac{1}{A^2}\left[\frac{B_{11}}{B} + \frac{A_{11}}{A} - \frac{A_1^2}{A^2}\right] = 8\pi p \tag{6}$$

$$\frac{1}{A^2}\left[\frac{A_1 B_1}{AB} + \frac{A_1 C_1}{AC} - \frac{B_1 C_1}{BC} - \frac{B_{11}}{B} - \frac{C_{11}}{C}\right] = 8\pi\varepsilon \tag{7}$$

3. Solution of field equations

Equation (4) and (5) lead to

$$\left(\frac{A_1}{A}\right)_1 - \frac{A_1}{A}\left(\frac{B_1}{B}+\frac{C_1}{C}\right) = \frac{B_1C_1}{BC} - \frac{C_{11}}{C} \tag{8}$$

From equations (5) and (6), we have

$$\frac{B_{11}}{B} - \frac{C_{11}}{C} = 0 \tag{9}$$

Equations (4) to (7) are four equations in five unknown A, B, C, ε and p. To get the deterministic model, we assume that universe is filled with barotropic perfect fluid distribution. This leads to $p = \gamma\,\varepsilon$ where $0 \le \gamma \le 1$. Now equations (4) and (7) after using barotropic perfect fluid condition ($p = \gamma\,\varepsilon$) lead to

$$(1-\gamma)\frac{A_1}{A}\left(\frac{B_1}{B}+\frac{C_1}{C}\right) + (1+\gamma)\frac{B_1C_1}{BC} + \gamma\left(\frac{B_{11}}{B}+\frac{C_{11}}{C}\right) = 0 \tag{10}$$

Equation (9) leads to

$$(CB_1 - BC_1)_1 = 0 \tag{11}$$

which leads to

$$C^2\left(\frac{B}{C}\right)_1 = K \quad \text{(constant)} \tag{12}$$

we assume that

$$BC = \mu, \quad B/C = \nu \tag{13}$$

Thus, we have

$$B^2 = \mu\,\nu, \quad C^2 = \mu/\nu \tag{14}$$

Using equations (13) and (14) in equation (12), we have

$$\frac{\nu_1}{\nu} = \frac{K}{\mu} \tag{15}$$

Using equations (13) and (15) in equation (10), we have

$$(1-\gamma)\frac{A_1}{A}\left(\frac{\mu_1}{\mu}\right) + \left(\frac{1-\gamma}{4}\right)\frac{\mu_1^2}{\mu^2} + \left(\frac{\gamma-1}{4}\right)\frac{K^2}{\mu^2} + \frac{\gamma\,\mu_{11}}{\mu} = 0 \tag{16}$$

which leads to

$$\frac{A_1}{A} + \frac{1}{4}\frac{\mu_1}{\mu} - \frac{K^2}{4\mu\mu_1} + \frac{\gamma}{1-\gamma}\frac{\mu_{11}}{\mu_1} = 0 \tag{17}$$

To get the deterministic solution, we assume that

$$K = 0$$

Thus equation (15) leads to

$$\frac{v_1}{v} = 0 \tag{18}$$

We, thus have

$$v = \text{constant} = l \quad \text{(say)} \tag{19}$$

Using equation (18) in equation (17), we have

$$\frac{A_1}{A} = -\frac{1}{4}\frac{\mu_1}{\mu} - \frac{\gamma}{1-\gamma}\frac{\mu_{11}}{\mu_1} \tag{20}$$

which leads to

$$\left(\frac{A_1}{A}\right)_1 = -\frac{1}{4}\frac{\mu_{11}}{\mu} + \frac{1}{4}\frac{\mu_1^2}{\mu^2} - \frac{\gamma}{1-\gamma}\frac{\mu_{111}}{\mu_1} + \frac{\gamma}{1-\gamma}\frac{\mu_{11}^2}{\mu_1^2} \tag{21}$$

Now using equations (18), (20) and (21) in equation (8), we have

$$\frac{d}{dx}\left(\frac{\mu_{11}}{\mu_1}\right) = \left(\frac{1+3\gamma}{4\gamma}\right)\frac{\mu_{11}}{\mu} \tag{22}$$

Let us assume that $\mu_1 = f(\mu)$. This leads to $\mu_{11} = ff'$ and $f' = \dfrac{df}{d\mu}$.

Now from equation (22), we have

$$f = M\mu^n + \beta \tag{23}$$

where $M = \dfrac{4\alpha\gamma}{1+7\gamma}$ (24)

$$n = \frac{1+7\gamma}{4\gamma} = \frac{\alpha}{M} \tag{25}$$

where α and β are constants of integration.

From equation (20), we have

$$A^2 = \frac{N^2 \mu^{-1/2}}{(M\mu^n + \beta)^{2\gamma/1-\gamma}} \tag{26}$$

where N is the constant of integration and

$$B^2 = \mu\nu = l\mu \tag{27}$$

$$C^2 = \frac{\mu}{\nu} = \frac{\mu}{l} \tag{28}$$

Hence the metric (1) reduces to the form

$$ds^2 = \frac{N^2 \mu^{-1/2}}{[M\mu^n + \beta]^{2\gamma/(1-\gamma)}} \left[\frac{dx^2}{d\mu^2} d\mu^2 - dt^2 \right] + l\mu\, dy^2 + \frac{\mu}{l} dz^2 \tag{29}$$

After suitable transformation of coordinates, the metric (29) leads to

$$ds^2 = \frac{N^2 X^{-1/2}}{[MX^n + \beta]^{2\gamma/(1-\gamma)}} \left[\frac{dX^2}{(MX^n + \beta)^2} - dT^2 \right] + X\, dY^2 + X\, dZ^2 \tag{30}$$

where $\mu = X,\ \frac{1}{\sqrt{l}} z = Z,\ \sqrt{l}\, y = Y,\ t = T.$

4. Some physical and geometrical features

The energy density (ρ), the isotropic pressure (p) forthe model (30) are given by

$$8\pi\rho = -\frac{\alpha}{1-\gamma} X^{n-3/2} (MX^n + \beta)^{\frac{1+\gamma}{1-\gamma}} \tag{31}$$

and

$$8\pi p = -\frac{\alpha\gamma}{1-\gamma} X^{n-3/2} (MX^n + \beta)^{\frac{1+\gamma}{1-\gamma}} \tag{32}$$

The components of conformal curvature tensor (C^{jk}_{hi}) are given by

$$C^{12}_{12} = \frac{(MX^n + \beta)^{\frac{1+\gamma}{1-\gamma}}}{24 X^{3/2}} \left[\left\{ \frac{4\gamma(n-1)\alpha + 3\alpha(1-\gamma) - 3M(1-\gamma)}{(1-\gamma)} \right\} X^n - 3\beta \right] \tag{33}$$

$$C^{13}_{13} = \frac{(MX^n + \beta)^{\frac{1+\gamma}{1-\gamma}}}{24 X^{3/2}} \left[\left\{ \frac{4\gamma(n-1)\alpha + 3\alpha(1-\gamma) - 3M(1-\gamma)}{(1-\gamma)} \right\} X^n - 3\beta \right] \tag{34}$$

$$C^{14}_{14} = \frac{(MX^{n} + \beta)^{\frac{1+\gamma}{1-\gamma}}}{12\, X^{3/2}} \left[-\left\{ \frac{4\,\gamma\,(n-1)\,\alpha + 3\alpha\,(1-\gamma) - 3M\,(1-\gamma)}{(1-\gamma)} \right\} X^{n} + 3\,\beta \right] \tag{35}$$

The reality conditions (i) $(\rho + p) > 0$, (ii) $(\rho + 3p) > 0$ given by Ellis [4] together lead to

$$\frac{\alpha}{1-\gamma}\,(1+\gamma) < 0$$

which leads to $\alpha < 0$ as $0 \le \gamma \le 1$.

The model (30) represents an universe of Petrov type I degenerate. For large value of X, the space-time is conformally flat.

5. Special cases

To get the deterministic value of μ, we assume that $n = 3$ which leads to $\gamma = 1/5$ and $n = 4$ leads to $\gamma = 1/9$. These are physically valid results for barotropic perfect fluid condition.

Case (i). For $n = 3$, we have

$$n = \frac{1 + 7\gamma}{4\,\gamma} = 3$$

which leads to $\gamma = 1/5$

and $M = \dfrac{4\,\alpha\,\gamma}{1 + 7\gamma} = \dfrac{\alpha}{3}$

Now the equation (23) leads to

$$\frac{d\mu}{dx} = \frac{\alpha}{3}\,\mu^{3} + \beta \tag{36}$$

which again leads to

$$\frac{d\mu}{\left(\mu^{3} + \dfrac{3\,\beta}{\alpha}\right)} = \frac{\alpha}{3}\,dx \tag{37}$$

which on integration leads to

$$\frac{1}{6\,N^{2}} \log \frac{(\mu + N)^{2}}{(\mu^{2} - N\mu + N^{2})} + \frac{1}{\sqrt{3}\,N^{2}} \tan^{-1}\left(\frac{2\mu - N}{\sqrt{3}\,N}\right) = \left(\frac{\alpha}{3}\right) x + m \tag{38}$$

where $N = \left(\frac{3\beta}{\alpha}\right)^{1/3}$

and m is constant of integration.

Thus we have

$$A^2 = N^2\left[\frac{1}{\mu\left(\frac{\alpha}{3}\mu^3+\beta\right)}\right]^{1/2} \tag{39}$$

$$B^2 = l\,\mu \tag{40}$$

$$C^2 = \mu/l \tag{41}$$

where μ is determined by (38).

Thus the metric (1) leads to

$$ds^2 = N^2\left[\frac{1}{X\left(\frac{\alpha}{3}X^3+\beta\right)}\right]^{1/2}\left[\frac{dX^2}{\left(\frac{\alpha}{3}X^3+\beta\right)^2}-dT^2\right]+X\,dY^2+X\,dZ^2 \tag{42}$$

where $\mu = X,\ \sqrt{l}\,y = Y,\ \frac{1}{\sqrt{l}}z = Z,\ t = T.$

The energy density (ρ), the isotropic pressure (p) for the model (42) are given by

$$8\pi\rho = -\frac{5}{4}\alpha\left[X\left(\frac{\alpha}{3}X^3+\beta\right)\right]^{-3/2} \tag{43}$$

and

$$8\pi p = -\frac{\alpha}{4}\left[X\left(\frac{\alpha}{3}X^3+\beta\right)\right]^{-3/2} \tag{44}$$

The components of conformal curvature tensor are given by

$$C^{12}_{12} = \frac{1}{8}\left(\frac{4}{3}\alpha X^3-\beta\right)\left[\frac{1}{X}\left(\frac{\alpha}{3}X^3+\beta\right)\right]^{-3/2} \tag{45}$$

$$C^{13}_{13} = \frac{1}{8}\left(\frac{4}{3}\alpha X^3-\beta\right)\left[\frac{1}{X}\left(\frac{\alpha}{3}X^3+\beta\right)\right]^{-3/2} \tag{46}$$

$$C^{14}_{14} = -\frac{1}{4}\left(\frac{4}{3}\alpha X^3-\beta\right)\left[\frac{1}{X}\left(\frac{\alpha}{3}X^3+\beta\right)\right]^{-3/2} \tag{47}$$

The reality conditions (i) $(\rho + p) > 0$, (ii) $\rho + 3p > 0$ given by Ellis [4] are satisfied when $\alpha < 0$ for the model (42). The energy density (ρ) is positive for $\alpha < 0$. The model (42) represents Petrov type I degenerate universe.

Case (ii). For $n = 4$, we have

$$n = \frac{1 + 7\gamma}{4\gamma} = 4$$

This leads to $\gamma = 1/9$

and $M = \dfrac{4\alpha\gamma}{1 + 7\gamma} = \dfrac{\alpha}{4}$ as $\gamma = \dfrac{1}{9}$.

Now equation (23) leads to

$$\frac{d\mu}{dx} = \frac{\alpha}{4}\mu^4 + \beta \tag{48}$$

which again leads to

$$\frac{d\mu}{\mu^4 + \dfrac{4\beta}{\alpha}} = \frac{\alpha}{4}dx \tag{49}$$

which on integration leads to

$$\frac{1}{4N^3}\log\frac{(\mu - N)}{(\mu + N)} - \frac{1}{2N^3}\tan^{-1}\frac{\mu}{N} = \frac{\alpha}{4}x + L \tag{50}$$

where $N = \left(-\dfrac{4\beta}{\alpha}\right)^{1/4}$, $\alpha < 0$, $\beta > 0$

and L is constant of integration.

Thus we have

$$A^2 = N^2\left[\frac{1}{\mu\left(\dfrac{\alpha}{4}\mu^4 + \beta\right)}\right]^{1/2} \tag{51}$$

$$B^2 = l\mu \tag{52}$$

$$C^2 = \mu/l \tag{53}$$

where μ is determined by equation (50).

Thus the metric (1) leads to

$$ds^2 = N^2\left[\frac{1}{X\left(\frac{\alpha}{4}X^4+\beta\right)}\right]^{1/2}\left[\frac{dX^2}{\left(\frac{\alpha}{4}X^4+\beta\right)^2} - dT^2\right] + X\,dY^2 + X\,dZ^2 \tag{54}$$

where $\mu = X,\ \sqrt{l}\,y = Y,\ \frac{1}{\sqrt{l}}z = Z,\ t = T.$

The energy density (ρ), the isotropic pressure (p) for the model (54) are given by

$$8\pi\rho = -\frac{9}{8}\alpha\left[X\left(\frac{\alpha}{4}X^4+\beta\right)\right]^{-3/2} \tag{55}$$

and

$$8\pi p = -\frac{1}{8}\alpha\left[X\left(\frac{\alpha}{4}X^4+\beta\right)\right]^{-3/2} \tag{56}$$

The components of conformal curvature tensor are given by

$$C^{12}_{12} = \frac{1}{8}\left(\frac{5}{4}\alpha X^4 - \beta\right)\left[\frac{1}{X}\left(\frac{\alpha}{4}X^4+\beta\right)\right]^{-3/2} \tag{57}$$

$$C^{13}_{13} = \frac{1}{8}\left(\frac{5}{4}\alpha X^4 - \beta\right)\left[\frac{1}{X}\left(\frac{\alpha}{4}X^4+\beta\right)\right]^{-3/2} \tag{58}$$

$$C^{14}_{14} = -\frac{1}{4}\left(\frac{5}{4}\alpha X^4 - \beta\right)\left[\frac{1}{X}\left(\frac{\alpha}{4}X^4+\beta\right)\right]^{-3/2} \tag{59}$$

The reality conditions (i) $\rho + p > 0$, (ii) $\rho + 3p > 0$ given by Ellis [4] are satisfied when $\alpha < 0$ for the model (54) also $\rho > 0$ when $\alpha < 0$. The space-time (54) in general represents Petrov type I degenerate.

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